EQUIVARIANT CELLULAR HOMOLOGY AND ITS APPLICATIONS

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In this work we develop the cellular equivariant homology functor and apply it to prove the equivariant Euler-Poincaré formula and the equivariant Lefschetz theorem.

1. Introduction

Let D be an arbitrary, small topologically enriched category. In this paper we develop a D-CW-homology functor which allows for easy computation of the *ordinary* D-equivariant homology defined by E. Dror Farjoun [1]. Our approach is a generalization of the G-CW-(co)homology functor constructed by S.J. Willson [14] for the case of Bredon homology with respect to a compact Lie group G.

Then we apply the D-CW-homology functor to obtain:

- The Equivariant Euler-Poincaré formula:

$$\chi^D(\underset{\sim}{X}) = \sum_{n=0}^{\infty}(-1)^n \widetilde{rk}_{\text{HS}}(H_n^D(\underset{\sim}{X};\mathcal{I})) \tag{1}$$

  This formula establishes a connection between the equivariant homology and the equivariant Euler characteristic; $\widetilde{rk}_{\text{HS}}(?)$ is a slight modification of the rank element originally introduced by Hattori [8] and Stallings [12]; see Remark 4.4 for its definition.
- The Equivariant Lefschetz theorem:

  Let $\underset{\sim}{X}$ be a triangulated D-space, $f : \underset{\sim}{X} \to \underset{\sim}{X}$ an equivariant map. If the equivariant Lefschetz number

$$\Lambda_D(f) = \sum_{n=0}^{\infty}(-1)^n \widetilde{tr}_{\text{HS}}(H_n^D(f;\mathcal{I})) \tag{2}$$

  is not equal to zero, then there are f-invariant orbits in $\underset{\sim}{X}$. Moreover, the orbit types of the invariant orbits may be recovered from $\Lambda_D(f)$; $\widetilde{tr}_{\text{HS}}(?)$ is a modification of the Hattori-Stallings trace similar to the above variation of the Hattori-Stallings rank; see Remark 4.9 for details.

2. Preliminaries

2.1. **D-spaces.** Let $\mathcal{T}$op denote the category of the compactly generated Hausdorff topological spaces. D is a fixed but arbitrary small category enriched over $\mathcal{T}$op. We work in the category $\mathcal{T}\text{op}^D$ of functors from D to $\mathcal{T}$op. The objects of this category are called topological diagrams or just D-spaces. The arrows in $\mathcal{T}\text{op}^D$ are natural transformations of functors or *equivariant maps.*

2.2. **D-homotopy.** An equivariant homotopy between two D-maps $f, g : \underset{\sim}{X} \to \underset{\sim}{Y}$, where $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are D-diagrams, is a D-map $H : \underset{\sim}{X} \times I \to \underset{\sim}{Y}$, where I denotes the constant D-space $I(d) = [0, 1]$. A *homotopy equivalence* $f : \underset{\sim}{X} \to \underset{\sim}{Y}$ is a map with a (two sided) D-homotopy inverse.

2.3. **D-orbits.** We recall now the central concept of the D-homotopy theory (introduced in [1],[3]) – that of D-orbit. A D-orbit is a D-space $T : D \to \mathcal{T}\mathrm{op}$, such that $\operatorname{colim}_D T = \{*\}$. F^d is a **free** D-orbit generated in $d \in \operatorname{obj}(D)$ if $\mathcal{T}\mathrm{op}^D \ni F^d = \hom_D(d, ?)$, i.e. $F^d(d') = \hom_D(d, d')$ and $F^d(d' \to d'')$ is given by the composition. Clearly F^d is a D-orbit. A D-space $\underset{\sim}{X}$ is called free if for any $s \in \operatorname{colim}_{\mathcal{C}} \underset{\sim}{X}$ the full orbit T_s lying over s is free.

2.4. **D-CW-complexes.** A *D-cell* is a D-space of the form $T \times e^n$, where T is a D-orbit and e^n is the standard n-cell. An *attaching map* of this D-cell to some D-space $\underset{\sim}{X}$ is a map $\phi : T \times \partial e^n \to \underset{\sim}{X}$.

A (*relative*) D-CW-complex $(\underset{\sim}{X}, \underset{\sim}{X}_{-1})$ is a D-space $\underset{\sim}{X}$ together with a filtration $\underset{\sim}{X}_{-1} \subset \underset{\sim}{X}_0 \subset \ldots \subset \underset{\sim}{X}_n \subset \underset{\sim}{X}_{n+1} \subset \ldots \subset \underset{\sim}{X} = \operatorname{colim}_n \underset{\sim}{X}_n$, such that $\underset{\sim}{X}_{n+1}$ is obtained from $\underset{\sim}{X}_n$ by attaching a set of n-dimensional D-cells. In other words one has a push-out diagram of D-spaces:

$$\begin{tikzcd} \coprod_i (T_i \times \partial e^n) \arrow[r, "\phi"] \arrow[d] & \underset{\sim}{X}_{n-1} \arrow[d] \\ \coprod_i (T_i \times e^n) \arrow[r, "\Phi"] & \underset{\sim}{X}_n \end{tikzcd}$$

If $\underset{\sim}{X}_{-1} = \varnothing$ we call the D-CW-complex *absolute*.

Let $\underset{\sim}{X}$ be a D-CW-complex. A D-subspace $\underset{\sim}{Y} \subset \underset{\sim}{X}$ is called the *cellular subspace* if $\underset{\sim}{Y}$ has a D-CW-structure such that each cell of $\underset{\sim}{Y}$ is also a cell of $\underset{\sim}{X}$.

2.5. **The category of orbits.** To each small category D we associate its *category of orbits* $\mathcal{O}_D$. This is the full topological sub-category of $\mathcal{T}\mathrm{op}^D$ generated by all D-orbits.

Usually $\mathcal{O}_D$ is not a small category. For example, if $D = J = (\bullet \longrightarrow \bullet)$, then $\mathcal{O}_J \cong \mathcal{T}\mathrm{op}$.

If $\mathcal{O}$ is a small subcategory of $\mathcal{O}_D$, then a D-CW-complex $\underset{\sim}{X}$ is of *type* $\mathcal{O}$ if any orbit T used in the construction of $\underset{\sim}{X}$ is also an element of $\operatorname{obj}(\mathcal{O})$.

In this paper we will be mostly interested in diagrams which are homotopy equivalent to finite D-CW-complexes. It will always be assumed that any finite D-CW-complex under consideration is of orbit type $\mathcal{O}$ for some fixed full topological subcategory $\mathcal{O}$ of $\mathcal{O}_D$ with finite amount of objects.

2.6. **Orbit point $(?)^{\mathcal{O}}$ and realization $|?|_D$ functors.** Suppose $\mathcal{O}$ is a *small* category of D-orbits. The orbit point functor $(?)^{\mathcal{O}} : \mathcal{T}\mathrm{op}^D \to \mathcal{T}\mathrm{op}^{\mathcal{O}^{\mathrm{op}}}$ is the generalization to the diagram case of Bredon's fixed point functor. For any D-space $\underset{\sim}{X}$ (usually of type $\mathcal{O}$) $(\underset{\sim}{X})^{\mathcal{O}}$ is an $\mathcal{O}^{\mathrm{op}}$ diagram such that $(\underset{\sim}{X})^{\mathcal{O}}(T) = \hom_D(T, \underset{\sim}{X})$ for all $T \in \operatorname{obj}(\mathcal{O})$ and the arrows of the diagram are induced by composition with the maps between orbits.

If $f : \underset{\sim}{X} \to \underset{\sim}{Y}$ is an equivariant map between two D-spaces, then there exist an $\mathcal{O}^{\mathrm{op}}$-equivariant map $f^{\mathcal{O}} : \underset{\sim}{X}^{\mathcal{O}} \to \underset{\sim}{Y}^{\mathcal{O}}$, which is obtained from f by the composition:

$$\underset{\sim}{X}^{\mathcal{O}}(T) = \hom_D(T, \underset{\sim}{X}) \ni g \xmapsto{f^{\mathcal{O}}} f \circ g \in \hom_D(T, \underset{\sim}{Y}) = \underset{\sim}{Y}^{\mathcal{O}}(T)$$

The fundamental property of the $(?)^{\mathcal{O}}$ functor is that for any D-space $\underset{\sim}{X}$ the $\mathcal{O}^{\mathrm{op}}$-space $(\underset{\sim}{X})^{\mathcal{O}}$ is $\mathcal{O}^{\mathrm{op}}$-free [1, 3.7].

Example 2.1. Let $\mathcal{O}$ be the full subcategory of $\mathcal{O}_D$ generated by the set of free orbits. Then $\mathcal{O}$ is isomorphic to D^{op} as a category and $(\underset{\sim}{X})^{\mathcal{O}} \cong \underset{\sim}{X}$ for any diagram $\underset{\sim}{X}$ (Yoneda's lemma). However, only free D-spaces are of orbit type $\mathcal{O}$.

Another easy case occurs when $\underset{\sim}{X}$ is a D-CW-space. We shall discuss it in the next section.

If $D = G$ is a group, then the orbit point functor coincides with the fixed points sets functor, which relates a space with an action of G to the diagram of subspaces fixed by subgroups of G.

There exists a left adjoint to $(?)^{\mathcal{O}}$. It is called *realization functor* $|?|_D$, since it takes an $\mathcal{O}^{\mathrm{op}}$-space and produces a D-space with the prescribed orbit point data (up to local weak equivalence). Realization functors in the group case have been constructed by A.D.Elmendorf [7] and have been generalized to the arbitrary diagram case by W.Dwyer and D.Kan [6].

2.7. **Equivariant Euler characteristic.** Let $\underset{\sim}{X}$ be a finite D-CW-complex. If $\underset{\sim}{X}$ is of type $\mathcal{O}$ for some category of orbits with a finite amount of objects, then $h\mathcal{O}$ will denote the homotopy category of $\mathcal{O}$, i.e. $\mathrm{obj}(h\mathcal{O}) = \mathrm{obj}(\mathcal{O})$ and $\mathrm{mor}_{h\mathcal{O}}(\underset{\sim}{T}_1, \underset{\sim}{T}_2)$ is the set of D-homotopy classes of maps.

Let $U(D, \mathcal{O}) = U(D) = \bigoplus_{T \in \mathrm{Iso}(\mathrm{obj}(h\mathcal{O}))} \mathbb{Z}$ be the free abelian group generated by the finite set of homotopy classes of orbits in $\mathcal{O}$.

Let $n(\underset{\sim}{X}, T, i)$ be the number of i-dimensional cells of $\underset{\sim}{X}$ of type T and put

$$n(\underset{\sim}{X}, T) = \sum_{i \geq 0} n(\underset{\sim}{X}, T, i).$$

We define the *equivariant Euler characteristic* $\chi^D(\underset{\sim}{X}) \in U(D)$ by the formula

$$\chi^D(\underset{\sim}{X}) = \sum_{[T] \in \mathrm{Iso}(h\mathcal{O})} n(\underset{\sim}{X}, T)[T].$$

Equivalently, the equivariant Euler characteristic is the Universal Additive Invariant [10, I.5] $(U(D), \chi^D)$ of the category of finite D-CW-complexes of type $\mathcal{O}$, where the cofibrations are relative D-CW-complexes.

3. Equivariant cellular homology

3.1. **$\mathcal{O}^{\mathrm{op}}$-$CW$–structure on the orbit point diagram of a D-CW–complex.** The construction of the equivariant (co)homology functor [1, 4.16] depends on a cellular decomposition of the orbit point diagram $\underset{\sim}{X}^{\mathcal{O}}$. In this section we show that if $\underset{\sim}{X}$ is a D-CW–complex, then $\underset{\sim}{X}^{\mathcal{O}}$ has a simple $\mathcal{O}^{\mathrm{op}}$-$CW$–structure.

For any orbit T the functor $\hom_D(T, ?) : \mathcal{T}\mathrm{op}^D \to \mathcal{T}\mathrm{op}$ commutes, obviously, with coproducts. This functor is a right adjoint, hence, it commutes with inverse limits, but it does not commute, in general, with pushouts. For example, if T_n denotes a J-diagram with n points mapped into one point, then $T_1 \coprod_{T_0} T_2 = T_3$. Applying the functor $\hom_J(T_2, ?)$, we obtain

$$\begin{aligned}\hom_J(T_2, T_1) \coprod_{\hom_J(T_2, T_0)} \hom_J(T_2, T_2) &= \{5 \text{ points}\} \neq \\ \hom_J(T_2, T_3) &= \{9 \text{ points}\}.\end{aligned}$$

Despite the example above, the functor $\hom_J(T_2, ?)$ commutes with pushouts that attach cells in D-CW-complexes.

Proposition 3.1. *Let $\underset{\sim}{X}$ be a D-CW-complex of orbit type $\mathcal{O}$, for some full, small subcategory $\mathcal{O}$ of the category of orbits. Then $\underset{\sim}{X}^{\mathcal{O}}$ has an $\mathcal{O}^{\mathrm{op}}$-$CW$-structure induced by the D-CW-structure of $\underset{\sim}{X}$ in the following sense: if $\underset{\sim}{X}_0 \subseteq \underset{\sim}{X}_1 \subseteq \cdots \subseteq \underset{\sim}{X}_n \subseteq \cdots \subseteq \underset{\sim}{X} = \mathrm{colim}_n \underset{\sim}{X}_n$ is a D-CW filtration of $\underset{\sim}{X}$, such that each $\underset{\sim}{X}_n$ is the push-out*

$$\begin{array}{ccc} \coprod_i T_i \times S^{n-1} & \xrightarrow{\phi} & \underset{\sim}{X}_{n-1} \\ \big\downarrow & & \big\downarrow{\scriptstyle i_n} \\ \coprod_i T_i \times D^n & \xrightarrow{\Phi} & \underset{\sim}{X}_n, \end{array} \tag{3}$$

then there exists an $\mathcal{O}^{\mathrm{op}}$-$CW$-filtration $\underset{\sim}{X}^{\mathcal{O}}_0 \subseteq \underset{\sim}{X}^{\mathcal{O}}_1 \subseteq \cdots \subseteq \underset{\sim}{X}^{\mathcal{O}}_n \subseteq \cdots \subseteq \underset{\sim}{X}^{\mathcal{O}} = \mathrm{colim}_n \underset{\sim}{X}^{\mathcal{O}}_n$, where $\underset{\sim}{X}^{\mathcal{O}}_n = (\underset{\sim}{X}_n)^{\mathcal{O}}$, and

$$\begin{array}{ccc} \coprod_i F^{T_i} \times S^{n-1} & \xrightarrow{\phi^{\mathcal{O}}} & \underset{\sim}{X}^{\mathcal{O}}_{n-1} \\ \big\downarrow & & \big\downarrow{\scriptstyle i^{\mathcal{O}}_n} \\ \coprod_i F^{T_i} \times D^n & \xrightarrow{\Phi^{\mathcal{O}}} & \underset{\sim}{X}^{\mathcal{O}}_n \end{array} \tag{4}$$

is a push-out square.

Proof. We proceed by induction on the skeleton of $\underset{\sim}{X}$. If $\underset{\sim}{X}_0 = \coprod T_i$, then

$$\begin{aligned} \underset{\sim}{X}^{\mathcal{O}}_0(T) = (\coprod T_i)^{\mathcal{O}}(T) = \hom_D(T, \coprod T_i) &= \coprod \hom_D(T, T_i) = \\ \coprod \hom_{\mathcal{O}^{\mathrm{op}}}(F^T, F^{T_i}) &= \coprod F^{T_i}(T). \end{aligned}$$

Hence the base of the induction.

Let us assume the proposition for $\underset{\sim}{X}_{n-1}$. We need to prove it for $\underset{\sim}{X}_n$. Applying the functor $(?)^{\mathcal{O}}$ on the commutative square 3, we obtain the commutative square 4. To complete the proof we need to show that 4 is a push-out. Direct limits of diagrams of functors are computed object by object, so it suffices to show that for each orbit $T \in \mathcal{O}$ the commutative diagram

$$\begin{array}{ccc} \coprod_i \hom(T, T_i) \times S^{n-1} & \xrightarrow{\hom(T,\phi)} & \hom(T, \underset{\sim}{X}_{n-1}) \\ \big\downarrow & & \big\downarrow{\scriptstyle \hom(T, i_n)} \\ \coprod_i \hom(T, T_i) \times D^n & \xrightarrow{\hom(T,\Phi)} & \hom(T, \underset{\sim}{X}_n) \end{array} \tag{5}$$

is a pushout of topological spaces.

First we show that the commutative square 5 is a pushout of sets and afterwards we notice that this is also a pushout in the category of compactly generated spaces $\mathcal{T}$op.

Upon application of the forgetful functor $U : \mathcal{T}\mathrm{op} \to \mathcal{S}\mathrm{ets}$ (which we do not write explicitly), each entry of the commutative square 5 equals the coproduct of the corresponding entries of the following squares (recall that $\hom(T, ?)$ commutes with coproducts).

$$\begin{array}{ccc}
\coprod_i \hom(T, T_i) \times S^{n-1} & \longrightarrow & \hom(T, \underset{\sim}{X}_{n-1}) \\
\downarrow & & \downarrow \\
\coprod_i \hom(T, T_i) \times S^{n-1} & \longrightarrow & \hom(T, \underset{\sim}{X}_n \setminus (\coprod_i T_i \times \overset{\circ}{D}{}^n))
\end{array}$$

$$\begin{array}{ccc}
\varnothing & \longrightarrow & \varnothing \\
\downarrow & & \downarrow \\
\hom(T, \coprod_i T_i \times \overset{\circ}{D}{}^n) & \longrightarrow & \hom(T, \coprod_i T_i \times \overset{\circ}{D}{}^n)
\end{array}$$

Both squares are pushouts of sets (in the first square vertical arrows are bijections and in the second horizontal arrows are bijections). The commutative square 5 is a pushout of sets, since pushouts commute with coproducts.

We work in the category of compactly generated Hausdorff spaces and a pushout in this category may be calculated as a pushout in the category of all Hausdorff spaces. But if the same pushout, calculated in the category of all topological spaces, appears to be a Hausdorff space, then it is a pushout in the subcategory of Hausdorff spaces. (See, e.g., Mac Lane [11, VII.8.2, V.9.2].) It remains to show that the square 5 is a push-out in the category of all topological spaces. We know that this is a pushout of sets, and it is routine to check that the space $\hom(T, \underset{\sim}{X}_n)$ (topologized as a subspace of $\times_{d \in \mathrm{obj}(D)} \hom(T(d), \underset{\sim}{X}_n(d))$) has the topology of colimit in the category of all topological spaces. $\square$

3.2. ***D*-*CW*-homology functor.** Equivariant homological algebra was introduced by Ch. Watts [13] and became a useful tool for treating various versions of equivariant homology. We refer to T. tom Dieck [4] for a modern account of the theory.

To define the equivariant *cellular* chain complex, take $\mathcal{C}_q(\underset{\sim}{X})$ to be the *free* $R\mathcal{O}^{\mathrm{op}}$-module, where to each q-dimensional cell of $\underset{\sim}{X}^{\mathcal{O}}$ of type F^T corresponds the direct summand $R(F^T) = R\hom_{\mathcal{O}}(?, T) = R\hom_{\mathcal{O}^{\mathrm{op}}}(T, ?)$ of $\mathcal{C}_q(\underset{\sim}{X})$. ($R(F^T)$ is a free $R\mathcal{O}^{\mathrm{op}}$-module in the sense of [4]).

Boundary maps are defined easily on the generators of free $R\mathcal{O}^{\mathrm{op}}$-modules $\mathcal{C}_q(\underset{\sim}{X})$. These generators correspond bijectively to generators of free R-modules $H_q((\mathrm{colim}_D\, \underset{\sim}{X})_q, (\mathrm{colim}_D\, \underset{\sim}{X})_{q-1}, R)$ which are isomorphic, by definition, to the CW-chain complexes of $\mathrm{colim}_D\, \underset{\sim}{X} \cong \mathrm{colim}_D\, \underset{\sim}{X}^{\mathcal{O}}$ [1, p. 111]. But for the ordinary CW-chains, boundary maps

$$H_q((\mathrm{colim}\, \underset{\sim}{X})_q, (\mathrm{colim}\, \underset{\sim}{X})_{q-1}, R) \xrightarrow{\partial_q} H_{q-1}((\mathrm{colim}\, \underset{\sim}{X})_{q-1}, (\mathrm{colim}\, \underset{\sim}{X})_{q-2}, R)$$

are defined as connecting homomorphisms in the long exact sequence of the triple $((\mathrm{colim}\, \underset{\sim}{X})_q, (\mathrm{colim}\, \underset{\sim}{X})_{q-1}, (\mathrm{colim}\, \underset{\sim}{X})_{q-2})$. Thus we can extend the definition uniquely to boundary maps in the equivariant cellular chain complex.

For any homotopy (co)functor $M\colon \mathcal{O} \to R\text{-}mod$ we define (co)chains with coefficients in M by

$$\mathcal{C}^{\mathcal{O}}_*(\underset{\sim}{X}, M) = M \otimes_{\mathcal{O}^{\mathrm{op}}} \mathcal{C}_*(\underset{\sim}{X}) \tag{6}$$

$$\mathcal{C}^*_{\mathcal{O}}(\underset{\sim}{X}, M) = \hom_{\mathcal{O}^{\mathrm{op}}}(\mathcal{C}_*(\underset{\sim}{X}), M). \tag{7}$$

The (co)homology $H^D_*(\underset{\sim}{X}; M)$ ($H^*_D(\underset{\sim}{X}; M)$) of $\underset{\sim}{X}$ with coefficients in M is the (co)homology of the (co)chain complex 6 (7).

3.3. **Isotropy ring $\mathcal{I}$.** S.J. Willson [14] has developed a universal coefficient system for the G-equivariant homology (where G is a compact Lie group). Let us generalize his approach to the coefficient systems for the classical D-homology theory. Suppose $\mathcal{O}$ is a small, full subcategory of the orbit category $\mathcal{O}_D$. If $\underset{\sim}{X}$ is a D-space of orbit type $\mathcal{O}$, then a coefficient system for the ordinary (co)homology is a homotopy (co)functor $M : \mathcal{O} \to (R - \mathrm{mod})$.

Definition 3.2. Let R be a commutative ring. An *isotropy ring* $\mathcal{I} = I_{\mathrm{D}}^{\mathrm{R},\mathcal{O}}$ is generated by $\mathrm{mor}(h\mathcal{O})$ as a free R-module. Define the multiplication on the generators by

$$fg = \begin{cases} f \circ g, & \text{if } \mathrm{codom}(g) = \mathrm{dom}(f) \\ 0, & \text{otherwise} \end{cases}$$

and extend the definition to the rest of the elements of $\mathcal{I}$ by linearity.

Proposition 3.3. *The category $\mathcal{M}$ of the left $\mathcal{I}$-modules which satisfy:*

$$\forall M \in \mathrm{obj}(\mathcal{M}),\ M = \bigoplus_{T \in \mathrm{obj}(h\mathcal{O})} 1_T M \tag{8}$$

(where $\{1_T M\}_{T \in \mathrm{obj}(h\mathcal{O})}$ are left R-modules) and the category of $R(h\mathcal{O})$-mod of functors from $h\mathcal{O}$ to the category of left R-modules are equivalent.

Proof. Let us define a pair of functors which induce the required equivalence:

$$\zeta : \mathcal{M} \leftrightarrows R(h\mathcal{O})\text{-}mod : \xi.$$

Let $M \in \mathrm{obj}(\mathcal{M})$, $T \in \mathrm{obj}(h\mathcal{O})$, then define

$$\zeta M(T) = 1_T M.$$

If $\mathrm{mor}(h\mathcal{O}) \ni f : T_1 \to T_2$, then define

$$\zeta M(f)(1_{T_1} m) = f 1_{T_1} m = (1_{T_2} f) 1_{T_1} m \in 1_{T_2} M.$$

Obviously the morphisms of the left $\mathcal{I}$ modules correspond to the natural transformations of the functors.

Given a $R(h\mathcal{O})$-module N, then we have

$$\xi N = \bigoplus_{T \in \mathrm{obj}(h\mathcal{O})} N(T), \text{as a left } R\text{-module.}$$

Define the left $\mathcal{I}$-module structure on ξN by $f(\dots, n, \dots) = (\dots, fn, \dots)$, where

$$N(\mathrm{codom}(f)) \ni fn = \begin{cases} f(n), & \text{if } n \in N(\mathrm{dom}(f)) \\ 0, & \text{otherwise} \end{cases}$$

Now it is clear that the defined functors provide the equivalence of the categories. □

Remark 3.4. The ring $\mathcal{I}$, considered as a left $\mathcal{I}$-module, is an object of $\mathcal{M}$, because $\mathcal{I} \cong \bigoplus_{T \in \mathrm{obj}(h\mathcal{O})} 1_T \mathcal{I}$ (as left R-modules) by the construction. But it also carries an obvious structure of the right $\mathcal{I}$ module, so does the $\zeta\mathcal{I}(T)$.

Remark 3.5. If $\mathrm{obj}(h\mathcal{O})$ is finite then the ring $\mathcal{I}$ has a two-sided identity element $1 = \sum_{T \in \mathrm{obj}(h\mathcal{O})} 1_T$, together with its decomposition into the sum of the orthogonal idempotents and the condition (8) is redundant.

Definition 3.6. The augmentation map $\varphi : \mathcal{I} \to \bigoplus_{T\in \mathrm{Iso}(\mathrm{obj}(h\mathcal{O}))} R$ is defined for any

$$\mathcal{I} \ni g = \sum_{T\in \mathrm{obj}(h\mathcal{O})} \sum_{f\in \mathrm{mor}(T,T)} r_f f + \sum_{h\in \mathrm{mor}(T_1,T_2), T_1\neq T_2} s_h h$$

(only a finite number of $r_f, s_h \in R$ are non equal to zero) to be

$$\varphi(g) = (\dots, \sum_{f\in \mathrm{mor}(T,T)} r_f, \dots) \in \bigoplus_{T\in \mathrm{Iso}(\mathrm{obj}(h\mathcal{O}))} R$$

Example 3.7. If the category $\mathcal{O}$ is a topological group, then the isotropy ring is just the group ring $\mathcal{I} = R[\mathcal{O}]$. The map φ in this case is the ordinary augmentation map: $\varphi(\sum r_i f_i) = \sum r_i \in R$.

Remark 3.8. The idempotents in $\mathcal{I}$ which correspond to the D-homotopy equivalent orbits are identified under φ. Clearly φ is an epimorphism of rings. Consider the abelinization functor $Ab : (Rings) \to \mathcal{A}b$ that maps a ring to its additive group divided by the commutator subgroup. Then $Ab(\varphi)$ is a homomorphism of abelian groups $Ab(\varphi) : Ab(\mathcal{I}) \to \bigoplus_{T\in \mathrm{Iso}(\mathrm{obj}(h\mathcal{O}))} R$. The last map will be used to obtain a generalization of the Euler-Poincaré formula.

4. Applications

Let $\underset{\sim}{X}$ be a finite D-CW-complex of type $\mathcal{O}$ for some orbit category $\mathcal{O}$ with $\mathrm{obj}(\mathcal{O})$ a *finite* set.

4.1. Equivariant Euler-Poincaré formula.

We recall that the equivariant Euler characteristic lies in the abelian group $U(D) \cong \bigoplus_{\mathrm{Iso}(\mathrm{obj}(h\mathcal{O}))} \mathbb{Z}$, so in order to apply the Hattori–Stallings machinery we need to choose a coefficient system for the equivariant homology such that the resulting chain complex and homology groups will be endowed with the module structure over some ring S which allows an epimorphism $\varepsilon : Ab(S) \longrightarrow U(D)$, to ensure that every possible value of the equivariant Euler characteristic is covered.

Our choice of the coefficients will be the isotropy ring $\mathcal{I} = I_D^{\mathbb{Z},\mathcal{O}}$ taken over itself as a left module. Then we take $\varepsilon = Ab(\varphi) \colon Ab(\mathcal{I}) \to \bigoplus_{\mathrm{Iso}(\mathrm{obj}(h\mathcal{O}))} \mathbb{Z} \cong U(D)$, which is an epimorphism by construction.

Lemma 4.1. *Let $\underset{\sim}{X}$ be a finite D-CW-complex. Suppose $\underset{\sim}{X}$ has n_q q-dimensional cells and $t_1 + \dots + t_s = n_q$, where t_i is the number of q-dimensional cells of the same homotopy type $T_i \in \mathrm{Iso}(\mathrm{obj}(h\mathcal{O}))$. Then $\mathcal{C}_q(\underset{\sim}{X}) \otimes_{\mathcal{O}} \zeta\mathcal{I} \cong \zeta\mathcal{I}(T_1)^{t_1} \oplus \dots \oplus \zeta\mathcal{I}(T_n)^{t_s}$ as a left $\mathbb{Z}$-module.*

Proof. Let $t_i = r_{i1} + \dots + r_{ik}$, where r_{ij} is the number of q-dimensional cells of type $T_{ij} \in \mathrm{obj}(\mathcal{O})$ of homotopy type T_i. By the construction of the equivariant homology, $\mathcal{C}_q(\underset{\sim}{X}) = \bigoplus_{i=1}^{s} (\oplus_{j=1}^{k} \mathbb{Z}(\hom_{\mathcal{O}}(?, T_{ij})^{r_{ij}}))$. The dual Yoneda isomorphism [9, p.74] implies that

$$\mathcal{C}_q(\underset{\sim}{X}) \otimes_{\mathcal{O}} \zeta\mathcal{I} \cong \bigoplus_{i=1}^{s} (\oplus_{j=1}^{k} \zeta\mathcal{I}(T_{ij})^{r_{ij}}) \cong \bigoplus_{i=1}^{s} \oplus_{j=1}^{k} (1_{T_{ij}}\mathcal{I})^{r_{ij}}.$$

If T_{ij_1} is isomorphic to T_{ij_2} in $h\mathcal{O}$ then there is an obvious isomorphism of the left $\mathbb{Z}$-modules and right $\mathcal{I}$-modules $1_{T_{ij_1}}\mathcal{I} \cong 1_{T_{ij_2}}\mathcal{I}$. Let us choose a representative T_i

of each isomorphism class of objects in $h\mathcal{O}$. Then

$$\mathcal{C}_q(\underset{\sim}{X}) \otimes_{\mathcal{O}} \zeta\mathcal{I} \cong \bigoplus_{i=1}^{s} (1_{T_i}\mathcal{I})^{(\sum_{j=1}^{k} r_{ij})} \cong \bigoplus_{i=1}^{s} (1_{T_i}\mathcal{I})^{t_i} \cong \bigoplus_{i=1}^{s} (\zeta\mathcal{I}(T_i))^{t_i}$$

□

Because of 3.4 the equivariant chain complex $\{\mathcal{C}_q(\underset{\sim}{X}) \otimes_{\mathcal{O}} \zeta\mathcal{I}\}_{q=0}^{\dim \underset{\sim}{X}}$ is a complex of *projective* right $\mathcal{I}$-modules and the equivariant homology is endowed with the right $\mathcal{I}$-module structure.

Notation: The Euler characteristic of an $\mathcal{I}$-differential complex with respect to $rk_{\mathrm{HS}}(?)$ is denoted by $\chi_{\mathrm{HS}}(?)$.

Proposition 4.2. *Let $K_* = \mathcal{C}_*(\underset{\sim}{X}) \otimes_{\mathcal{O}} \zeta\mathcal{I}$ be a right $\mathcal{I}$-complex, then $\chi^D(\underset{\sim}{X}) = Ab(\varphi)(\chi_{\mathrm{HS}}(K_*))$, whenever the left side is defined.*

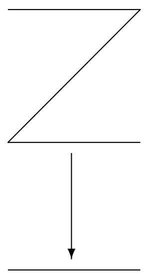
Proof. It is easy to see that $rk_{\mathrm{HS}}(1_T\mathcal{I}) = 1_T \in Ab(\mathcal{I})$. Lemma 4.1 together with 3.8 completes the proof. □

Now we combine 4.2 with the additivity properties of the Hattori-Stallings rank and obtain the following

Theorem 4.3. $\chi^D(\underset{\sim}{X}) = Ab(\varphi)(\sum_{n=0}^{\infty}(-1)^n rk_{\mathrm{HS}} H_n^D(\underset{\sim}{X};\zeta\mathcal{I}))$*, whenever the left side is defined.*

Remark 4.4. One can denote $\widetilde{rk}_{\mathrm{HS}}(?) = Ab(\varphi)(rk_{\mathrm{HS}}(?))$ to obtain from the last theorem the main formula (1).

Example 4.5. Consider the following J-diagram:

$\underset{\sim}{Z} =$ 

The diagram $\underset{\sim}{Z}$ has two 0-cells of type $T_2 = \left[\begin{smallmatrix} \cdot\,\cdot \\ \downarrow \\ \cdot \end{smallmatrix}\right]$ and one 1-cell of type $T_3 = \left[\begin{smallmatrix} \cdot\,\cdot\,\cdot \\ \downarrow \\ \cdot \end{smallmatrix}\right]$, hence,

$$\chi^J(\underset{\sim}{Z}) = 2\left[\begin{smallmatrix} \cdot\,\cdot \\ \downarrow \\ \cdot \end{smallmatrix}\right] - \left[\begin{smallmatrix} \cdot\,\cdot\,\cdot \\ \downarrow \\ \cdot \end{smallmatrix}\right].$$

In this case a possible choice for the category $\mathcal{O}$ is the full subcategory of $\mathcal{O}_D$ generated by the two objects T_2 and T_3. The cellular chain complex tensored with the coefficients $\mathcal{I} = \mathcal{I}_J^{\mathbb{Z},\mathcal{O}}$ becomes:

$$\cdots \to 0 \to 1_{T_3}\mathcal{I} \xrightarrow{\partial_1} (1_{T_2}\mathcal{I})^2$$

and $\partial_1 = 0$ from orbit type considerations. $U(J) = \mathbb{Z} \oplus \mathbb{Z}$ in our case; $H_0^J(\underset{\sim}{Z}, \mathcal{I}) = (1_{T_2}\mathcal{I})^2$, $H_1^J(\underset{\sim}{Z}, \mathcal{I}) = 1_{T_3}\mathcal{I}$ are right $\mathcal{I}$ - modules. Hence, $\chi^J(\underset{\sim}{Z}) = (2,0) - (0,1) = (2,-1)$, as expected.

Let us calculate, for comparison, the J-equivariant homology of $\underset{\sim}{Z}$ with the constant coefficients $\mathbb{Z}^{\mathcal{O}}(T) = \mathbb{Z}$ for any T in $\mathrm{obj}(\mathcal{O})$ and $\mathbb{Z}^{\mathcal{O}}(f) = id_{\mathbb{Z}}$ for any $f \in \mathrm{mor}(\mathcal{O})$. $H_i^J(\underset{\sim}{Z}, \mathbb{Z}^{\mathcal{O}}) = H_i(\mathrm{colim}_J\, \underset{\sim}{Z}, \mathbb{Z})$. Then $\mathrm{colim}_J\, \underset{\sim}{Z} = I = [0,1]$ and

$$H_i^J(\underset{\sim}{Z}, \mathbb{Z}^{\mathcal{O}}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0 \\ 0, & \text{otherwise} \end{cases}$$

Thus, the constant diagram of coefficients $\mathbb{Z}^{\mathcal{O}}$ is inappropriate for the Euler-Poincaré formula.

4.2. **Equivariant Lefschetz theorem.** Using the cellular equivariant homology functor we can now prove a version of the equivariant Lefschetz theorem.

Some Lefschetz type results in the equivariant setting may be obtained already by applying the ordinary Lefschetz theorem: consider an equivariant map $f : \underset{\sim}{X} \to \underset{\sim}{X}$, where $\underset{\sim}{X}$ is a diagram over the small category D, then if the Lefschetz number $\Lambda(\mathrm{colim}_D\, \underset{\sim}{X}) \neq 0$, there are f-invariant D-orbits in $\underset{\sim}{X}$. However, the advantage of using the equivariant homology and equivariant Lefschetz number $\Lambda_D(\underset{\sim}{X}) \in U(D)$ is that we obtain specific information about the orbit type of the invariant orbit.

First, we give a technical

Definition 4.6. A D-CW-complex $\underset{\sim}{X}$ is called a *triangulated* D-space if the natural CW-structure of $\mathrm{colim}\, \underset{\sim}{X}$ also triangulates $\mathrm{colim}\, \underset{\sim}{X}$.

The following lemma will be used in the proof of the equivariant Lefschetz theorem.

Lemma 4.7. *Let $\underset{\sim}{X}$ be a triangulated diagram, then for any refinement Y of the triangulation of $\mathrm{colim}\, \underset{\sim}{X}$, there exists a D-CW-complex $\underset{\sim}{X}'$, such that $\underset{\sim}{X}'$ is D-homeomorphic to $\underset{\sim}{X}$ and $\mathrm{colim}\, \underset{\sim}{X}' = Y$ (as triangulated spaces). $\underset{\sim}{X}'$ will be called the refinement of $\underset{\sim}{X}$.*

Proof. We proceed by induction on the dimension n of the simplices of Y. The base of the induction is clear. Suppose that we have managed to provide the refinement of $\underset{\sim}{X}$ for any simplex of Y up to dimension $n-1$. Consider a new simplex Δ of dimension n in the triangulation of Y. It lies in some old simplex of $\mathrm{colim}\, \underset{\sim}{X}$: $\Delta \in \Delta'$. Then consider the pull-back:

$$\lim \begin{pmatrix} & \underset{\sim}{X} \\ & \downarrow \\ \mathring{\Delta} \hookrightarrow & \mathrm{colim}_D\, \underset{\sim}{X} \end{pmatrix} = T \times \mathring{\Delta},$$

where T is the orbit that lies over the baricenter of Δ'.

Take $T \times \Delta$ to be an n-cell of the new D-$CW-$complex $\underset{\sim}{X}'$. The attachment map $\phi\colon T \times \partial\Delta \to \underset{\sim}{X}'_{n-1} \subset \underset{\sim}{X}$ is just the inclusion on the orbits that lie over points of the boundary which are internal with respect to Δ'; for the orbits over boundary points of Δ', take ϕ to be the restriction of the attachment map of the simplex $T \times \Delta'$ in the triangulated D-CW-complex $\underset{\sim}{X}'$. Continuing in the same way for the rest of the n-simplices of Y completes the induction step.

By the construction, our D-CW-complex $\underset{\sim}{X}'$ has the same underlying topological diagram as $\underset{\sim}{X}$, therefore they are D-homeomorphic. $\square$

Definition 4.8. Let $f : \underset{\sim}{X} \to \underset{\sim}{X}$ be a map of a finite triangulated D-space $\underset{\sim}{X}$ of orbit type $\mathcal{O}$, where $\mathcal{O}$ is an orbit category containing a finite number n of objects.

If $\mathcal{I} = \mathcal{I}_D^{\mathbb{Z},\mathcal{O}}$, then the *equivariant Lefschetz number* of f is

$$U(D) \ni (\lambda_1, \dots, \lambda_n) = \Lambda_D(f) = Ab(\varphi)(\sum_{k=0}^{\infty} (-1)^k \mathrm{tr}_{\mathrm{HS}}(H_k^D(f;\mathcal{I}))).$$

Remark 4.9. If we denote $\widetilde{tr}_{\mathrm{HS}}(?) = Ab(\varphi)(rk_{\mathrm{HS}}(?))$, then the definition above coincides with Formula (2) for the Lefschetz number.

Theorem 4.10. *If $\underset{\sim}{X}$ is a finite triangulated diagram over D with a D-map $f : \underset{\sim}{X} \to \underset{\sim}{X}$ and $\Lambda_D(f) = (\lambda_1, \dots, \lambda_n) \in U(D)$ is the Lefschetz number of f, then for each m, if $\lambda_m \neq 0$, there exists an f-invariant orbit of type T_m in $\underset{\sim}{X}$.*

Proof. A simplex in $\operatorname{colim} \underset{\sim}{X}$ will be called of type T if the overlaying orbit is of type T in $\underset{\sim}{X}$. Suppose that there are no invariant orbits of type T_m. This is equivalent to the condition that the map induced on $\operatorname{colim} \underset{\sim}{X}$ has no fixed points in the simplices of type T_m.

Since $\underset{\sim}{X}$ is a finite triangulated diagram, $\operatorname{colim} \underset{\sim}{X}$ is a finite triangulated space, hence it is a compact metric space. If there are no fixed points of type T_m, then there exists a refinement Y of the triangulation such that if Δ is a simplex of type T_m in Y, $\Delta \cap (\operatorname{colim} f)(\Delta) = \varnothing$.

Consider the refinement $\underset{\sim}{X}'$ of $\underset{\sim}{X}$, which exists by Lemma 4.7. Since $\underset{\sim}{X}' \cong \underset{\sim}{X}$, $H_*^D(\underset{\sim}{X}';\mathcal{I}) = H_*^D(\underset{\sim}{X};\mathcal{I})$, $\Lambda(f') = \Lambda(f)$, where $f' : \underset{\sim}{X}' \to \underset{\sim}{X}'$ is equal to f, $\lambda'_m = \lambda_m$. We will show now that $\lambda'_m = 0$.

$$\begin{aligned} \Lambda_D(f') &= Ab(\varphi)(\sum_{k=0}^{\infty} (-1)^k \mathrm{tr}_{\mathrm{HS}}(H_k^D(f';\mathcal{I}))) \\ &= Ab(\varphi)(\sum_{k=0}^{\infty} (-1)^k \mathrm{tr}_{\mathrm{HS}}(\mathcal{C}_k^{\mathcal{O}}(f';\mathcal{I}))), \end{aligned}$$

where $\mathcal{C}_k^{\mathcal{O}}(f';\mathcal{I})$ is the map induced by f on the chains $\mathcal{C}_k^{\mathcal{O}}(\underset{\sim}{X};\mathcal{I}) = \mathcal{C}_k(\underset{\sim}{X}) \otimes_{\mathcal{O}} \mathcal{I} = (1_{T_1}\mathcal{I})^{t_1} \oplus \dots \oplus (1_{T_n}\mathcal{I})^{t_n}$ as an $\mathcal{I}$-module. Because of the property: $\Delta \cap (\operatorname{colim} f)(\Delta) = \varnothing$ for *any* simplex Δ of type T_m, the induced map on $\mathcal{C}_k^{\mathcal{O}}(\underset{\sim}{X};\mathcal{I})$ will take the generator 1_{T_m} corresponding to Δ outside the submodule $1_{T_m}\mathcal{I}$, that it generates. Then the m-th entry of $Ab(\varphi)(\sum_{k=0}^{\infty}(-1)^k \mathrm{tr}_{\mathrm{HS}}(\mathcal{C}_k^{\mathcal{O}}(f';\mathcal{I})))$ will be zero. This is true for all k, hence $\lambda_m = 0$, in contadiction to the initial assumption. Therefore there must exist invariant orbits of type T_m $\square$

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